

Trans-Planckian Issue in the Milne Universe

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The “trans-Planckian” challenge in cosmology appears when we trace the present physical wavelengths of fluctuations backwards in time. They become smaller and smaller until crossing the Planck scale where conventional QFT is challenged, so that unknown ultraviolet physics may be traced in the observable cosmological fluctuations. Usually this issue is addressed in the inflationary context, but trans-Planckian reasoning is much broader. We examine this logic in a simple example of scalar quantum field theory in the expanding and contracting Milne universes, where wavelengths of the eigenmodes are red- or blue-shifted. Trans-Planckian modifications of QFT should result in a UV-dependent VeV of the energy momentum tensor of a scalar field in the Milne universe. On the other hand, the Milne universe is another coordinate systems of flat Minkowski space-time, and the covariant energy momentum tensor should be the same (but vacuum-dependent) in different coordinates of flat space time. We explicitly demonstrate that in conventional QFT the energy momentum tensor, choosing the adiabatic vacuum, is identical to zero in Minkowski coordinates, and remains zero in the contracting Milne universe (due to non-trivial cancellations of contributions from particles which appear in the accelerating frame and from vacuum polarization there). In contrast to this, the trans-Planckian modification of the energy momentum tensor is not motivated. We provide a similar argument for the expanding Milne universe, where the energy momentum tensor in the conformal vacuum is non-zero. Similar arguments are applicable for other cosmological models where the curvature is much lower than Planckian which leads to conflicts with trans-Planckian considerations.

I. INTRODUCTION

The quantum theory of cosmological fluctuations generated during the inflationary stage of the very early universe describes the time evolution of the scalar eigenmodes $\phi_k(t)e^{i\vec{k}\cdot\vec{x}}$. The physical momentum of the eigenmodes $\vec{p} = \frac{\vec{k}}{a}$ is red-shifted in an expanding FRW universe with increasing scalar factor $a(t)$. If one takes a certain wavelength (of cosmological fluctuations) today and evolves it backwards in time, due to expansion this scale will shrink to smaller and smaller values and at one point will become smaller than the Planckian length. In other words, there are length scales visible today that have been below Planck length at some point in the past. This effect is especially dramatic during inflation, where the scalar factor increases exponentially. Ultimately, at some instance the physical momentum becomes equal to the Planckian mass scale M_p . After this point, the quantum field theory approach to fluctuations in an expanding universe (say, during inflation) should be replaced by a theory incorporating quantum gravity (say, string theory) which is not yet available. We found the earliest written traces of the trans-Planckian problem in [1]. The trans-Planckian challenge was articulated in [2], and discussed in many papers, see e.g [3].

Suppose that trans-Planckian effects alter the result of the conventional QFT at inflation. Despite an unclear notion about the microscopic theory of the effect, there is, however, a convenient phenomenological encoding for it in terms of Bogolyubov coefficients [4]. Indeed, as far

as the physical momentum of the mode is below M_p , QFT is applicable. Instead of the vacuum being the positive frequency eigenmode $f_k(\tau) = \frac{1}{2\omega_k}e^{-i\omega_k\tau}$, one can use the Bogolyubov coefficients A_k, B_k to describe the mode function of the initial state

$$f_k(\tau) \rightarrow A_k f_k(\tau) + B_k f_k^*(\tau), \quad |A_k|^2 - |B_k|^2 = 1. \quad (1)$$

UV physics, if any, is encoded in the B_k . The trans-Planckian effect looks pretty universal for any expanding FRW universe. Consider the Milne universe which is a

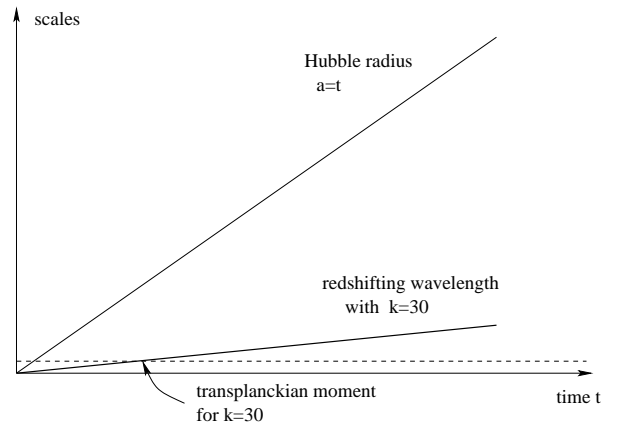


FIG. 1: Redshift of the wavelength $\lambda = \frac{2\pi}{k}a(t)$ in the Milne universe. The horizontal dashed line represents the Planck scale l_p . A given wavelength for example $k = 30$ crosses the Planckian scale at its trans-Planckian moment t_k .

hyperbolic space with FRW type metric

$$ds^2 = dt^2 - a(t)^2 (dr^2 + \sinh^2 r (d\theta^2 + \sin^2 \theta d\phi^2)) , \quad (2)$$

where $a(t) = t$ is the scalar factor. This is the co-moving coordinate system of the kinematic Milne model, which represents a medium of probe particles without gravity freely moving from the origin in empty space-time.

In this model, physical wavelengths are red-shifted, so the reasoning for a trans-Planckian effect should also be applied here, see Figure 1. However, the Milne universe is in fact another coordinate system of flat Minkowski space-time, where QFT can be treated analytically in great details, see e.g. [5]. Therefore QFT in the Milne universe can be used as a convenient ground to address the trans-Planckian effect. We will focus on the VeV of the energy-momentum tensor of a test scalar field, $\langle T_\mu^\nu \rangle$, which can be calculated in both coordinate systems of the flat space-time: in the usual Minkowski coordinates, and in the Milne coordinates (2). For a given choice of vacuum the answer for the covariant energy-momentum tensor $\langle T_\mu^\nu \rangle$ should be the same. However, for the calculation of $\langle T_\mu^\nu \rangle$ in the Milne coordinates, trans-Planckian effects (in terms of A_k and B_k) can be included and can alter the result. We will consider this as a test to the trans-Planckian prescription.

Traditionally, the trans-Planckian problem is considered in the context of an expanding universe, where A_k and B_k are inherited from the past. However, we can also consider the trans-Planckian problem for a contracting universe, where the horizon is still much bigger than Planckian size while the wavelengths are already blue-shifted below the Planckian scale. This takes place in the contracting phase of the Milne universe (similar to the Figure 1 but with reverse time direction). For higher momenta ($k \gg 1$), the “trans-Planckian” moment of time where a given wavelength crosses the Planckian scale occurs at time $t \gg t_p \sim 10^{-42}$ sec, and where we expect QFT in flat space-time to be valid.

Therefore we can calculate the VeV of the energy-momentum tensor at the “trans-Planckian” time in Minkowski coordinates which obviously vanishes

$$\langle T_\mu^\nu \rangle = 0 . \quad (3)$$

Our goal is to calculate $\langle T_\mu^\nu \rangle$ in the contracting Milne universe (2) with and without trans-Planckian contribution and compare it to the correct result (3). The calculation of $\langle T_\mu^\nu \rangle$ in the contracting Milne universe is technically easier than that in the expanding Milne universe (although still quite tedious). Therefore we first consider the problem in the contracting Milne universe where methods of calculations will be introduced, and then extend the results for the expanding Milne universe. We credit [6] where $\langle T_\mu^\nu \rangle_{1/2}$ for the spin 1/2 field in the contracting Milne universe was calculated, and we will extend its method to the case of the scalar field, and the book [7], where $\langle T_\mu^\nu \rangle$ for a scalar field in the contracting Milne universe was calculated by a different method.

We also consider $\langle T_\mu^\nu \rangle$ in the expanding Milne universe. Since the time t of the Milne coordinates and \tilde{t} of the Minkowski coordinates are connected by non-linear transformations, the vacuum choice for the expanding Milne universe is different from that in the contracting Milne universe. This is related to the choice of the conformal vacuum vs. adiabatic vacuum [5]. As a result, $\langle T_\mu^\nu \rangle$ in the Minkowski coordinates for this vacuum is non-zero and corresponds to an integration over the thermal spectrum of particles seen by an accelerating observer. On the other hand in an expanding Milne universe, in addition to this VeV, we may include a potential trans-Planckian contribution to A_k, B_k , which will alter the expected result.

The plan of this note is as follows. Section 2 contains a short introduction to the Milne universe. Section 3 is devoted to the general QFT in the FRW type universes, including the Milne universe. Section 4 gives a brief outline of the calculation of $\langle T_\mu^\nu \rangle$ in contracting Milne universe, while calculational details are collected in the appendix. Section 5 contains the extension of the results to the expanding Milne universe. In section 6 we discuss the challenge to the trans-Planckian challenge.

II. THE MILNE UNIVERSE

The metric of the Milne universe is a FRW type metric (2) or written in conformal coordinates

$$ds^2 = a(\eta)^2 (d\eta^2 - dr^2 - \sinh^2 r (d\theta^2 + \sin^2 \theta d\phi^2)) , \quad (4)$$

$$a(\eta) = e^\eta ,$$

where η is the conformal time. For the Milne metric the curvature tensor vanishes $R^\mu_{\nu\sigma\rho} = 0$, so it covers a portion of flat space-time. It is related to Minkowski space-time

$$ds^2 = d\tilde{t}^2 - d\tilde{r}^2 - \tilde{r}^2 (d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\tilde{\phi}^2) , \quad (5)$$

by a coordinate transformation

$$\begin{aligned} \tilde{t} &= t \cosh r, & \tilde{r} &= t \sinh r, \\ \tilde{\theta} &= \theta, & \tilde{\phi} &= \phi, \end{aligned} \quad (6)$$

covering the patch $\tilde{t}^2 - \tilde{r}^2 > 0$. The upper cone $t > 0$ corresponds to an expanding universe, the lower one $t < 0$ corresponds to the contracting universe, see Figure 2.

The conformal properties of the Milne universe are important for the choice of the vacuum, which is related to global time-like Killing vectors. Indeed, there is a useful theorem [5, 8]: If for two conformally related conformally flat space-times M_1, M_2 with M_2 flat there exists a diffeomorphism between a global Cauchy hypersurface σ_1 of M_1 and a global Cauchy hypersurface σ_2 of M_2 , then there exists also a correspondence between the global time-like Killing vector fields of M_1 and M_2 .

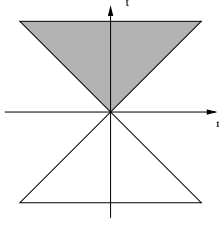


FIG. 2: The patches of Minkowski space covered by the Milne metric. Upper (shadow) patch corresponds to the expanding universe, while the lower (empty) patch corresponds to the contracting universe

Taking the Milne Universe as M_1 , it was found [8] (through mapping both space-times to the Einstein Universe) that these conditions are fulfilled when taking M_2 to be Rindler space with metric

$$ds^2 = e^{2a\xi} (d\eta^2 - d\xi^2) - dy^2 - dz^2, \quad (7)$$

which possesses two global time-like Killing vector fields

$$\begin{aligned} & \partial_\eta, \\ & e^{-a\xi} \cosh(a\eta) \partial_\eta - e^{-a\xi} \sinh(a\eta) \partial_\xi, \end{aligned} \quad (8)$$

corresponding to the conformal vacuum and the adiabatic vacuum.

So we can deduce that there are also two global time-like Killing vector fields for the Milne Universe, one of them defining the conformal vacuum, and the other the adiabatic vacuum. The corresponding conformal Killing vector fields in the Milne metric (4) are

$$\begin{aligned} & \Sigma_\mu^{(1)} : \partial_\eta, \\ & \Sigma_\mu^{(2)} : e^{-\eta} \cosh r \partial_\eta - e^{-\eta} \sinh r \partial_r, \end{aligned} \quad (9)$$

where η, r are now the coordinates of the metric (4).

We are faced with the choice which vacuum to take. It turns out that natural choice of vacuum for the contracting Milne universe is the adiabatic vacuum, associated with the Killing vector $\Sigma_\mu^{(2)}$. The adiabatic vacuum also corresponds to the usual vacuum in the Minkowski coordinates. Indeed, this Killing vector $\Sigma_\mu^{(2)}$ is nothing but $\partial_{\tilde{t}}$. The conformal vacuum associated with the other Killing vector $\Sigma_\mu^{(1)}$ is the natural choice of vacuum for the expanding Milne universe.

III. QUANTUM FIELD THEORY IN FRW SPACE-TIMES

Let us recall the basics of the QFT of a free massive real scalar field in a FRW background. The equation of motion for the scalar field is given by

$$\left(\square + m^2 - \frac{R}{6} \right) \phi = 0, \quad (10)$$

where $\square = D_\mu D^\mu$, m is the mass of the scalar field and R is the Ricci scalar. Using conformal time in the FRW universe and making the Ansatz

$$\phi(x^\mu) = a(\eta)^{-1} \int d\mu(\lambda, l, m) u_\lambda(\eta) \Psi_J(\vec{x}), \quad (11)$$

where Ψ_J are eigenfunctions of the spatial Laplacian $\Delta^{(3)} \Psi_J = \lambda^2 \Psi_J$ with quantum numbers $J = \{\lambda, l, m\}$ and $d\mu(J)$ is the measure over the quantum numbers, we obtain the following equation for the modes u_λ

$$\ddot{u}_\lambda + \omega^2 u_\lambda = 0, \quad \omega^2 = \lambda^2 + m^2 a^2, \quad (12)$$

where $\dot{u}_\lambda = \partial_\eta u_\lambda$ is the derivative with respect to conformal time. We quantize the field

$$\begin{aligned} \phi(x^\mu) = \frac{1}{\sqrt{2}} \int \frac{d\mu(J)}{a(\eta)} & \left(u_\lambda(\eta) \Psi_J(\vec{x}) a_J \right. \\ & \left. + u_\lambda^*(\eta) \Psi_J^*(\vec{x}) a_J^\dagger \right), \end{aligned} \quad (13)$$

where a_J^\dagger, a_J are the creation and annihilation operators for particles and antiparticles respectively. Using orthogonality relations for the eigenmodes of the spatial Laplacian it can be shown that the 00-component of the normal ordered energy momentum tensor for a scalar field is given by

$$\langle 0 | : T_0^0 : | 0 \rangle = \frac{1}{\pi^2 a^4} \int_0^\infty d\lambda \lambda^2 \omega s_\lambda, \quad (14)$$

where

$$s_\lambda = \frac{1}{2\omega} \left(|\dot{u}_\lambda|^2 + \omega^2 |u_\lambda|^2 - \omega \right). \quad (15)$$

Now we have to specify the vacuum $|0\rangle$ and the eigenmodes u_λ . The general solution to Equation (12) is

$$u_\lambda = c_1 H_{i\lambda}^{(1)}(\mu) + c_2 H_{i\lambda}^{(2)}(\mu), \quad (16)$$

where we defined $\mu = ma$ with $a = t = e^\eta$. $H_\nu^{(1,2)}(z)$ are the Hankel functions.

Now consider the contracting Milne universe. Initial conditions shall be defined at $t \rightarrow \infty$. For $\mu \rightarrow \infty$ we find a normalized positive energy solution

$$u_\lambda = \frac{\sqrt{\pi}}{2} e^{\frac{\pi}{2}\lambda} H_{i\lambda}^{(2)}(\mu). \quad (17)$$

The correct normalization follows from $(\phi_\lambda, \phi_\nu) = - \int_\Sigma d\Sigma^\mu \sqrt{-g_\Sigma} (\phi_\lambda \partial_\mu \phi_\nu^* - \phi_\nu^* \partial_\mu \phi_\lambda) \stackrel{!}{=} \delta_{\lambda,\nu}$ which translates into $u_\lambda \partial_t u_\nu^* - u_\nu^* \partial_t u_\lambda = i$ with Σ^μ being a time-like conformal Killing vector field orthogonal to the 3-surface of integration [9].

The choice (17) of the eigenmode corresponds to the adiabatic vacuum. Physically, the adiabatic vacuum corresponds to a vacuum which comes closest to being Minkowski, i.e. it should become Minkowski in the limit

of a very slowly changing geometry. In general this vacuum is associated with the WKB-type mode solutions of the equation of motion

$$u_\lambda = \frac{1}{\sqrt{2W_\lambda}} e^{-i \int^\eta d\eta' W_\lambda(\eta')}, \quad (18)$$

where W satisfies the non-linear equation

$$W_\lambda(\eta)^2 = \omega^2 - \frac{1}{2} \left(\frac{\ddot{W}_\lambda}{W_\lambda} - \frac{3}{2} \frac{\dot{W}_\lambda^2}{W_\lambda^2} \right). \quad (19)$$

Taking the limit of slowly varying scale factor a or rather equivalently the limit of large t (or η), we can approximate $\omega = me^\eta = W_\lambda$, so that the adiabatic approximation to the WKB-type solution is given by

$$u_\lambda = \frac{1}{\sqrt{2me^\eta}} e^{-ime^\eta}, \quad (20)$$

which corresponds to the large η limit of the eigenmodes (17). Thus, the adiabatic vacuum $|0_A\rangle$ of the contracting Milne Universe is the same as the Minkowski vacuum [5].

Next, we consider the expanding Milne universe. Initial conditions shall be defined at $\eta = -\infty$ ($t = 0$). We have to select the solution

$$u_\lambda \rightarrow v_\lambda = \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{\sinh \pi \lambda}} J_{-i\lambda}(\mu), \quad (21)$$

where $J_{i\lambda}(\mu)$ is the Bessel function. Its asymptotic form at $\eta \rightarrow -\infty$ is given by the normalized positive frequency solution $\frac{1}{\sqrt{2\lambda}} e^{-i\lambda\eta}$. This choice of eigenmode corresponds to the conformal vacuum.

Indeed, the conformal vacuum is obtained by performing a conformal transformation of the metric $g_{\mu\nu}$ to the metric $\tilde{g}_{\mu\nu}$, $g_{\mu\nu} = \Omega^2(x) \tilde{g}_{\mu\nu}$ which changes the equation of motion (10) to $\tilde{\square} \tilde{\Phi} = 0$, where $\tilde{\Phi} = \Omega^{-1} \Phi$. The vacuum state associated with the modes \tilde{u}_λ of $\tilde{\Phi}$ corresponds to the conformal vacuum $|0_C\rangle$.

It is instructive to compare the vacua of the contracting (adiabatic) and expanding (conformal) Milne universe. We can express the orthogonal set v_λ of normalized eigenfunctions of the expanding universe (21) in terms of the set of eigenfunctions u_λ of the contracting universe (17) introducing Bogolyubov coefficients

$$u_\lambda(\eta) = \alpha_\lambda v_\lambda(\eta) + \beta_\lambda v_\lambda^*(\eta). \quad (22)$$

Using the relation between the Bessel and the Hankel functions (see (37)) we find

$$\alpha_\lambda = \frac{e^{\frac{\pi}{2}\lambda}}{\sqrt{2 \sinh \pi \lambda}}, \quad \beta_\lambda = \frac{e^{-\frac{\pi}{2}\lambda}}{\sqrt{2 \sinh \pi \lambda}}. \quad (23)$$

In particular, this means that the conformal vacuum in Minkowski space-time corresponds to excitations of states related to the usual Minkowski adiabatic vacuum. In other words, in Minkowski coordinates $\langle 0_A | T_\mu^\nu | 0_A \rangle = 0$ but $\langle 0_C | T_\mu^\nu | 0_C \rangle \neq 0$.

IV. $\langle 0_A | T_\mu^{\nu(\text{Milne})} | 0_A \rangle$ IN CONTRACTING MILNE UNIVERSE

In this section we outline the calculation of the energy momentum tensor $\langle 0_A | T_\mu^{\nu(\text{Milne})} | 0_A \rangle$, in the contracting Milne universe. The starting point is expression (14) where we have to substitute the solution (17).

Formally the expression (14) for the mode functions in the Milne universe is divergent and needs to be regularized. Normal ordering in (14) takes care of the divergence coming from the zero point energy, but the energy momentum tensor in a curved space-time features more divergences which are attributed to vacuum polarization. In FRW type space-times it is most convenient to use the regularization method of Zel'dovich and Starobinsky [10], which we have adopted. The result is to replace s_λ in (14) by $s_\lambda - s_2 - s_4$ with (see (45) in the Appendix)

$$\begin{aligned} s_2 &= \frac{1}{16} \left(\frac{\dot{\omega}}{\omega^2} \right)^2, \\ s_4 &= -\frac{3}{256} \left(\frac{\dot{\omega}}{\omega^2} \right)^4 - \frac{1}{32} \frac{\dot{\omega}}{\omega^3} \frac{\partial}{\partial \eta} \left[\frac{1}{\omega} \frac{\partial}{\partial \eta} \left(\frac{\dot{\omega}}{\omega^2} \right) \right] \\ &\quad + \frac{1}{64} \left[\frac{1}{\omega} \frac{\partial}{\partial \eta} \left(\frac{\dot{\omega}}{\omega^2} \right) \right]^2. \end{aligned} \quad (24)$$

We illustrate the calculation for the energy density $\rho = T_{00}$, the other components of $T_{\mu\nu}$ can be either calculated similarly or from ρ , energy conservation $T_{\mu;\nu}^\mu$ and the vanishing of the conformal anomaly $\langle 0 | T_\mu^\mu | 0 \rangle = 0$.

For the renormalized energy density we have

$$T_0^{0(\text{ren})} = \rho_{\text{vac}} - \rho_0 - \rho_1 - \rho_2, \quad (25)$$

where

$$\begin{aligned} \rho_{\text{vac}} &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi^2 a^4} \int_0^\infty d\lambda \lambda^2 \left(|i\dot{u}_\lambda|^2 + \omega^2 |u_\lambda|^2 \right) e^{-\epsilon\lambda}, \\ \rho_0 &= \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi^2 a^4} \int_0^\infty d\lambda \lambda^2 \omega e^{-\epsilon\lambda}, \\ \rho_{1,2} &= \frac{1}{\pi^2 a^4} \int_0^\infty d\lambda \lambda^2 \omega s_{2,4}. \end{aligned} \quad (26)$$

The first two terms ρ_{vac} and ρ_0 are divergent, but after regularization they contain finite contributions. The finite part ρ_{vac} can be interpreted as a contribution of particles, seen by the comoving observer in contracting Milne universe. Terms $\rho_{1,2}$ are finite, and together with the finite part of ρ_0 can be interpreted as the vacuum polarization seen by the comoving observer.

In order to extract divergences in ρ_{vac} and ρ_0 , we introduce the regularizer $e^{-\epsilon\lambda}$ with small dimensionless parameter ϵ . At the end of our calculations the final answer will not be dependent on it and we can send ϵ to zero. This technical trick is borrowed from [11].

The results of calculations of the integrals (26) detailed in the appendix are the following

$$\rho_{\text{vac}} = \frac{1}{\pi^2 a^4} \left(\frac{3}{\epsilon^4} + \frac{\mu^2}{4\epsilon^2} + \frac{\mu^4}{16} \log \frac{\epsilon\mu}{2} \right)$$

$$+ \frac{\gamma\mu^4}{16} + \frac{\mu^4}{64} + \frac{\mu^2}{48} + \frac{1}{240} \Big), \quad (27)$$

$$\rho_0 = \frac{1}{\pi^2 a^4} \left(\frac{3}{\epsilon^4} + \frac{\mu^2}{4\epsilon^2} + \frac{\mu^4}{16} \log \frac{\epsilon\mu}{2} \right. \\ \left. + \frac{\gamma\mu^4}{16} + \frac{\mu^4}{64} \right), \quad (28)$$

$$\rho_1 = \frac{1}{\pi^2 a^4} \frac{1}{240}, \quad \rho_2 = \frac{1}{\pi^2 a^4} \frac{\mu^2}{48}. \quad (29)$$

In the net result (25) all divergences are canceled. However, all finite parts are also canceled so that we end up with

$$\langle 0_A | T_{0(\text{ren})}^{0(\text{Milne})} | 0_A \rangle = 0. \quad (30)$$

All other components of the energy momentum tensor are also zero.

V. $\langle 0_C | T_{\mu}^{\nu(\text{Milne})} | 0_C \rangle$ IN EXPANDING MILNE UNIVERSE

In this section we calculate the energy momentum tensor $\langle 0_C | T_{\mu}^{\nu(\text{Milne})} | 0_C \rangle$ in the expanding Milne universe. Again, we use the formula (14) where we substitute the solution (21).

Let us calculate energy density $\rho = \langle 0_C | T_{00}^{(\text{Milne})} | 0_C \rangle$. We have an integral expression similar to (14), but with the eigenfunctions v_{λ} instead of u_{λ} . In principle, we can apply the method of the previous section to this case, including regularizing with $e^{-\epsilon\lambda}$ and extracting divergences. It is easier, however, to use relationship (22) between v_{λ} and u_{λ} . Then we obtain

$$\langle 0_C | T_0^{0(\text{Milne})} | 0_C \rangle = \langle 0_A | T_0^{0(\text{Milne})} | 0_A \rangle \\ + \frac{1}{\pi^2 a^4} \int_0^{\infty} d\lambda \lambda^2 \omega |\beta_{\lambda}|^2 (2s_{\lambda} + 1) \\ + \Delta, \quad (31)$$

where s_{λ} , constructed from $|u_{\lambda}|$, is defined in (15). The residual term Δ is defined in (52) and is constructed from $u_{\lambda}^2, u_{\lambda}^{*2}$. Before regularization all the divergences are in the first term $\langle 0_A | T_{00}^{(\text{Milne})} | 0_A \rangle$.

Expression (31) becomes transparent for large values of t (or η), where $s_{\lambda} \rightarrow 1$, $\Delta \rightarrow 0$. Regularizing expression (31) is reduced to regularizing the first term which we performed in the previous section. Using (23) for β_{λ} , we have the final result

$$\langle 0_C | T_{0(\text{ren})}^{0(\text{Milne})} | 0_C \rangle = \frac{1}{\pi^2 a^4} \int_0^{\infty} d\lambda \lambda^2 \omega \frac{1}{e^{2\pi\lambda} - 1}. \quad (32)$$

As expected, in expanding Milne universe energy density is non-zero due to the choice of the conformal vacuum.

VI. DISCUSSION: CHALLENGE TO THE TRANS-PLANCKIAN CHALLENGE

We will discuss all aspects of the UV physics which may emerge in the cosmological models, for instance, the impact of the horizon [14] on the trans-Planckian issue.

In an expanding/contracting flat universe, a given wavelength of the oscillator's eigenmode $e^{i\vec{k}\vec{x}}$ is red-shifted/blue-shifted. The Milne universe has hyperbolic three dimensional spatial slicing. The spatial eigenmode of this hyperbolic space is described by the function (35) (see Appendix). For simplicity we consider the high-frequency modes with $\lambda \gg 1$. In this limit the eigenmode (35) is reduced to simple standing waves $\sim \cos(\lambda r)$. For these modes we can use the intuitive red-shifting/blue-shifting picture.

In this section we will discuss how the results (30), (32) will be changed if we apply the trans-Planckian prescription for the eigenmodes with the wavelengths which were or will be below the Planckian scale in the contracting/expanding Milne universe.

In the contracting universe a given wavelength is blue-shifted and will be shorter than the Planckian scale at its "trans-Planckian" moment t_k . If trans-Planckian effects work in this model, then the value $\langle 0_A | T_{00(\text{ren})}^{(\text{Milne})} | 0_A \rangle$ will departure from zero. However, the moment t_k is much bigger than the Planckian time. The moment t_k is not special for observers in the usual Minkowski coordinate system, where $\langle 0_A | T_{00(\text{ren})}^{(\text{Mink})} | 0_A \rangle$ remains always zero. We conclude that the trans-Planckian effect does not emerge in the contracting Milne universe.

Now, let us apply the trans-Planckian prescription to the calculation of $\langle 0_C | T_{00(\text{ren})}^{(\text{Milne})} | 0_C \rangle$ for the expanding universe. We simply alter the eigenmode $u_{\lambda}(\eta)$ at the moment η_k , according to (1). Then the result (30) will be changed to

$$\langle 0_C | T_{0(\text{trans})}^{0(\text{Milne})} | 0_C \rangle = \frac{1}{\pi^2 a^4} \int_0^{\infty} d\lambda \lambda^2 \omega |\beta_{\lambda} + B_{\lambda}|^2. \quad (33)$$

However, observers in the usual Minkowski coordinate system should not see any effects of B_{λ} . No trans-Planckian effects emerge in the expanding Milne universe.

It is clear that our consideration of the trans-Planckian issue in contracting/expanding cosmologies is much more general than the simple example of the Milne universe. Consider, for example, the expanding anisotropic Kasner universe

$$ds^2 = dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2, \quad (34)$$

with $\sum p_i = \sum p_i^2 = 1$, so that one of the p_i -s is non-positive. Suppose space is stretching in two directions while shrinking in the third direction, $p_3 < 0$. The component of the momentum k_3 of the quantum modes $e^{ik_3 z}$ associated with this third direction is blueshifting (while the universe as a whole is expanding), and at some

point passes the Planckian scale. It is paradoxical to encounter quantum gravity effects in an expanding Kasner universe! Notice the special combination of parameters $p_1 = p_2 = 0, p_3 = 1$. In this case the Kasner metric (34) is the product of the two-dimensional Milne universe and R^2 and can be transformed to the Minkowski space-time, and we have another example where the trans-Planckian issue evaporates. The resolution of the paradox: the criterion for the UV (string) physics to become important is having large values of the curvature terms $C_{\mu\nu\rho\sigma}^{\mu\nu\rho\sigma}, R_{\mu\nu}^{\mu\nu}, R^2 \sim 1/l_p^4$ but not coordinate effects.

We can compare the situation with another apparent paradox of inflation, using the “trans-Planckian” value of the inflaton field $\phi \gg M_p$. There are no restrictions to use the large ϕ values, the physical restrictions are related to the energy density $V \ll M_p^4$ (where the curvature is sub-Planckian).

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Appendix

In this appendix we describe properties of the eigenfunctions $u_\lambda \Psi_J(\vec{x})$ of (10) and present some details of the renormalization procedure for the 0–0 component of the energy momentum tensor in the conformal and adiabatic vacuum.

Properties of Eigenfunctions

For the hyperbolic case of the open universe (as in the case of the Milne metric), $J = \lambda, l, m$. Ignoring non-normalizable super-horizon modes, we have $0 \leq \lambda \leq \infty$, $l = 0, 1, 2, \dots$, $m = -l, \dots, +l$. The explicit form of the normalized space-dependent part of the eigenfunction is

$$\Psi_J(\vec{x}) = \frac{1}{\sqrt{\sinh r}} \frac{\Gamma(i\lambda + l + 1)}{|\Gamma(i\lambda)|} P_{i\lambda-1/2}^{-l-1/2}(\cosh r) Y_{lm}(\theta, \phi), \quad (35)$$

where P_ν^μ are the associated Legendre polynomials, and Y_{lm} are the spherical harmonics. Let us focus on the high frequency ($\lambda \gg 1$) asymptotic of $\Psi_J(\vec{x})$. The asymptotic properties of $P_{i\lambda-1/2}^{-l-1/2}(\cosh r)$ imply [12]

$$P_{i\lambda-1/2}^{-l-1/2}(\cosh r) \sim \cos(\lambda r). \quad (36)$$

The time-dependent part of the eigenfunctions u_λ is reduced to the solution of the Bessel equation (12).

The two modes u_λ , corresponding to the adiabatic vacuum, and v_λ , corresponding to the conformal vacuum are related through

$$v_\lambda = \frac{e^{\frac{\pi}{2}\lambda}}{\sqrt{2 \sinh \pi \lambda}} u_\lambda + \frac{e^{-\frac{\pi}{2}\lambda}}{\sqrt{2 \sinh \pi \lambda}} u_\lambda^*, \quad (37)$$

which can be obtained by using the relation between the Bessel and the Hankel functions for $\lambda, \mu \in \mathbb{R}$

$$J_{-i\lambda}(\mu) = \frac{1}{2} \left((H_{i\lambda}^{(2)}(\mu))^* + e^{\pi\lambda} H_{i\lambda}^{(2)}(\mu) \right). \quad (38)$$

This defines the coefficients $\alpha_\lambda, \beta_\lambda$ in (23).

Renormalizing $\langle 0_A | T_{00} | 0_A \rangle$

To compute the energy momentum tensor, we plug the solutions (17) of the mode equation (12) into the definition (14) of $\langle 0_C | T_{00} | 0_C \rangle$. Making use of the following properties of the Hankel functions [13]

$$\begin{aligned} (H_\nu^{(1)}(x))^* &= H_{\nu^*}^{(2)}(x), \\ H_{-\nu}^{(1)}(z) &= e^{i\pi} H_\nu^{(1)}(z), \\ \frac{2\nu}{z} H_\nu^{(i)}(z) &= H_{\nu-1}^{(i)}(z) + H_{\nu+1}^{(i)}(z), \end{aligned} \quad (39)$$

we find

$$\begin{aligned} |\dot{u}_\lambda|^2 &= \frac{\pi}{4} \left[\frac{\mu^2}{2} (H_{i\lambda+1}^{(1)} H_{i\lambda+1}^{(2)} + H_{i\lambda-1}^{(1)} H_{i\lambda-1}^{(2)}) \right. \\ &\quad \left. + \lambda^2 H_{i\lambda}^{(1)} H_{i\lambda}^{(2)} \right], \\ |u_\lambda|^2 &= \frac{\pi}{4} H_{i\lambda}^{(1)} H_{i\lambda}^{(2)}. \end{aligned} \quad (40)$$

Now we apply

$$H_\nu^{(1)} H_\nu^{(2)} = \frac{4}{\pi^2} \int_0^\infty dx K_0(2\mu \sinh x) (e^{2\nu x} + e^{-2\nu x}), \quad (41)$$

where $K_0(z)$ is the MacDonald function, and introduce the regularizing factor $e^{-\epsilon\lambda}$ to get

$$\begin{aligned} \rho_{\text{vac}} &= \frac{1}{2\pi^2 a^4} \int_0^\infty d\lambda \lambda^2 (|\dot{u}_\lambda|^2 + \omega^2 |u_\lambda|^2) \\ &= \frac{2}{\pi^3 a^4} \int_0^\infty dx K_0(2\mu \sinh x) \\ &\quad \times \left[\mu^2 \cosh(x)^2 \int_0^\infty d\lambda \lambda^2 \cos(2\lambda x) e^{-\epsilon\lambda} \right. \\ &\quad \left. + \int_0^\infty d\lambda \lambda^4 \cos(2\lambda x) e^{-\epsilon\lambda} \right] \\ &= \frac{2}{\pi^3 a^4} \int_0^\infty dx K_0(2\mu \sinh x) \\ &\quad \times \left[\mu^2 \cosh(x)^2 \underbrace{\epsilon \frac{\epsilon^2 - 12x^2}{(\epsilon^2 + 4x^2)^3}}_{=:f} \right. \\ &\quad \left. + 12 \epsilon \underbrace{\frac{\epsilon^4 - 40\epsilon^2 x^2 + 80x^4}{(\epsilon^2 + 4x^2)^5}}_{=:h} \right]. \end{aligned} \quad (42)$$

Multiplying the fractions f, h by a power of x^n will result in a zero contribution for $n \geq 3$ and $n \geq 5$ respectively. Therefore we expand $K_0(2\mu \sinh x) \cosh^2 x$ to $O(x^3)$ and $K_0(2\mu \sinh x)$ to order $O(x^5)$ and perform the integrations to obtain

$$\rho_{\text{vac}} = \frac{1}{\pi^2 a^4} \left(\frac{3}{\epsilon^4} + \frac{\mu^2}{4\epsilon^2} + \frac{\mu^4}{16} \log \frac{\epsilon}{2} + \frac{\mu^4 \log \mu}{16} + \frac{\gamma \mu^4}{16} + \frac{\mu^4}{64} + \frac{\mu^2}{48} + \frac{1}{240} \right), \quad (43)$$

The energy momentum tensor contains divergences even after subtracting the vacuum energy ρ_0 which calculates to

$$\begin{aligned} \rho_0 &= \frac{1}{\pi^2 a^4} \int_0^\infty d\lambda \lambda^2 \frac{\omega}{2} e^{-\epsilon \lambda} \\ &= \frac{1}{\pi^2 a^4} \\ &\quad \times \left(\frac{3}{\epsilon^4} + \frac{\mu^2}{4\epsilon^2} + \frac{\mu^4}{16} \log \frac{\epsilon}{2} + \frac{\mu^4 \log \mu}{16} + \frac{\gamma \mu^4}{16} + \frac{\mu^4}{64} \right). \end{aligned} \quad (44)$$

To deal with the remaining divergences, we employ the Zel'dovich-Starobinsky regularization scheme [10] which amounts to introducing the variables $s_\lambda = |\beta_\lambda|^2$ and $u_\lambda - i v_\lambda = \pm 2\alpha_\lambda \beta_\lambda e^{-2i \int_{\eta_0}^\eta d\eta \omega}$ obeying

$$\begin{aligned} \dot{s}_\lambda &= \frac{1}{2} \frac{\dot{\omega}}{\omega} u_\lambda, \\ \dot{v}_\lambda &= 2\omega u_\lambda, \\ \dot{u}_\lambda &= \frac{\dot{\omega}}{\omega} (1 \pm 2s_\lambda) - 2\omega v_\lambda. \end{aligned} \quad (45)$$

We set $\lambda \rightarrow h\lambda, m \rightarrow hm, \omega \rightarrow h\omega$, expand $s_\lambda, u_\lambda, v_\lambda$ in a series

$$s_{h\lambda} = \sum_{\ell=1} \frac{1}{h^\ell} s_\ell, \quad u_{h\lambda} = \sum_{\ell=1} \frac{1}{h^\ell} u_\ell, \quad v_{h\lambda} = \sum_{\ell=1} \frac{1}{h^\ell} v_\ell, \quad (46)$$

and plug them into (45) to find after some algebra

$$\begin{aligned} s_4 &= -\frac{3}{256} \left(\frac{\dot{\omega}}{\omega^2} \right)^4 - \frac{1}{32} \frac{\dot{\omega}}{\omega^3} \frac{\partial}{\partial \eta} \left[\frac{1}{\omega} \frac{\partial}{\partial \eta} \left(\frac{\dot{\omega}}{\omega^2} \right) \right] \\ &\quad + \frac{1}{64} \left[\frac{1}{\omega} \frac{\partial}{\partial \eta} \left(\frac{\dot{\omega}}{\omega^2} \right) \right]^2, \quad s_2 = \frac{1}{16} \left(\frac{\dot{\omega}}{\omega^2} \right)^2, \end{aligned} \quad (47)$$

with all other $s_\ell = 0$. Now we can easily calculate

$$\rho_1 = \frac{1}{\pi^2 a^4} \int_0^\infty d\lambda \lambda^2 \omega s_2 = \frac{1}{\pi^2 a^2} \frac{\mu^2}{48},$$

$$\rho_2 = \frac{1}{\pi^2 a^4} \int_0^\infty d\lambda \lambda^2 \omega s_4 = \frac{1}{\pi^2 a^2} \frac{1}{240}. \quad (48)$$

Adding up all terms gives

$$\langle 0_A | T_0^{0(\text{ren})} | 0_A \rangle = \rho_{\text{vac}} - \rho_0 - \rho_1 - \rho_2 = 0. \quad (49)$$

Renormalizing $\langle 0_C | T_{00} | 0_C \rangle$

In order to compute $\langle 0_C | T_{00} | 0_C \rangle$ we plug the conformal eigenmodes (21) into (15), express the modes v_λ, v_λ^* in terms of the modes u_λ, u_λ^* (17) and find after some algebra

$$\begin{aligned} \bar{s}_\lambda &= \frac{1}{2\omega} (|\dot{v}_\lambda|^2 + \omega^2 |v_\lambda|^2 - \omega) \\ &= s_\lambda + \frac{e^{-\pi\lambda}}{\sinh \pi\lambda} \left(s_\lambda + \frac{1}{2} \right) + \tilde{\Delta}, \end{aligned} \quad (50)$$

where we can identify the second term as $|\beta_\lambda|^2(2s_\lambda + 1)$ and

$$\tilde{\Delta} = \frac{1}{4\omega \sinh \pi\lambda} (\dot{u}_\lambda^2 + \omega^2 u_\lambda^2 + \dot{u}_\lambda^{*2} + \omega^2 u_\lambda^{*2}). \quad (51)$$

This gives rise to expression (31) for $\langle 0_C | T_0^0 | 0_C \rangle$ with

$$\Delta = \frac{1}{\pi^2 a^4} \int d\lambda \lambda^2 \omega \tilde{\Delta}. \quad (52)$$

At late times, the integrand $\tilde{\Delta}$ is rapidly oscillating so that $\lim_{t \rightarrow \infty} \Delta = 0$ as can be seen by using the large argument asymptotics of the Hankel functions

$$\begin{aligned} \lim_{\mu \rightarrow \infty} H^{(1)}(\mu) &= \sqrt{\frac{2}{\pi\mu}} e^{i(\mu - \frac{1}{2}\nu\pi - \frac{\pi}{4})}, \\ \lim_{\mu \rightarrow \infty} H^{(2)}(\mu) &= \sqrt{\frac{2}{\pi\mu}} e^{-i(\mu - \frac{1}{2}\nu\pi - \frac{\pi}{4})}. \end{aligned} \quad (53)$$

The late time behaviour of $(2s_\lambda + 1) \rightarrow 1$ can be easily seen from (40) using the asymptotic behaviour of the Hankel functions (53).

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